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The chain lemma for biquaternion algebras

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ABSTRACT

Any two decompositions of a biquaternion algebra over a field F into a sum of two quaternion algebras can be connected by a chain of decompositions such that any two neighboring decompositions are $(a,b)+(c,d)$ and $(ac,b)+(c,bd)$ for some $a,b,c,d \in F^*$. A similar result is established for decompositions of a biquaternion algebra into a sum of three quaternions if F has no cubic extension.

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Let A be a biquaternion algebra (i.e. a tensor product of two quaternion algebras) over a field F of characteristic different from 2. A decomposition of A into a tensor product of two quaternion algebras is not unique, and there is no canonical one. However, it turns out that any two decompositions of A can be connected by a chain of decompositions in which neighboring ones do not differ "too much". In fact in this note we prove an analogue of the chain lemma (see, for instance [L], where it is called "Common Slot Theorem") for a quaternion algebra.

So let $A = D_1 + D_1' = D_2 + D_2'$ be two decompositions of A into a sum of two quaternion algebras (the signs $=$ and $+$ will always mean equality and addition in the Brauer group of F). Dimension count shows that this means

$$A \simeq D_1 \otimes_F D_1' \simeq D_2 \otimes_F D_2'.$$

We call these decompositions equal if $D_1 = D_2$ and $D_1' = D_2'$, and simply-equivalent if there exist elements $x, y, a, c \in F$ such that $D_{1\,F(\sqrt{a})} = D'_{1\,F(\sqrt{c})} = 0$ and

$$D_2 = D_1 + \left(a, x^2 - acy^2\right), \qquad D_2' = D_1' + \left(c, x^2 - acy^2\right). \tag{$*$}$$

Notice that, since $(ac, x^2 - acy^2) = 0$, we have $D_1 + D_1' = D_2 + D_2'$ as soon as the equalities $(*)$ hold. We say that two decompositions of A are equivalent if they can be connected by a chain of

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decompositions in such a way that every two neighboring decompositions in this chain are simply-equivalent. The following result justifies this definition.

Proposition 1. *Any two biquaternion decompositions of A are equivalent to one another, and can be connected by a chain of length* 3*. Moreover, this bound is strict, i.e. in general two decompositions of A cannot be connected by a chain of length* 2*.*

Proof. Let $A = (a_1, b_1) + (c_1, d_1) = (a_2, b_2) + (c_2, d_2)$ be two decompositions of A. Assume first that the algebras (a_1, b_1) and (a_2, b_2) have a common splitting quadratic extension. In this case we may suppose that $a_1 = a_2$. Hence $(c_1, d_1) + (c_2, d_2) = (a_1, b_1 b_2)$, so (c_1, d_1) and (c_2, d_2) have a common splitting quadratic extension [A]. Therefore, we may suppose that $c_1 = c_2$. This implies that $(a_1, b_1 b_2) = (c_1, d_1 d_2)$. Denote this algebra by Q. We have $Q_{F(\sqrt{a_1})} = Q_{F(\sqrt{c_1})} = 0$. It is easy to verify that $Q \simeq (a_1, x^2 - a_1 c_1 y^2)$ for some $x, y \in F$. Hence

$$(a_2, b_2) = (a_1, b_1) + (a_1, b_1 b_2) = (a_1, b_1) + \bigl(a_1, x^2 - a_1 c_1 y^2\bigr),$$

and

$$(c_2, d_2) = (c_1, d_1) + (c_1, d_1 d_2) = (c_1, d_1) + \bigl(a_1, x^2 - a_1 c_1 y^2\bigr).$$

In particular, the decompositions $(a_1, b_1) + (c_1, d_1)$ and $(a_2, b_2) + (c_2, d_2)$ are simply-equivalent. This implies that in the general case it suffices to find $x_1, y_1, x_2, y_2 \in F$ such that the algebras $(a_1, b_1(x_1^2 - a_1 c_1 y_1^2))$ and $(a_2, b_2(x_2^2 - a_2 c_2 y_2^2))$ have a common quadratic splitting extension. This certainly will be the case if the form

$$\bigl\langle a_1, b_1\bigl(x_1^2 - a_1 c_1 y_1^2\bigr), -a_2, -b_2\bigl(x_2^2 - a_2 c_2 y_2^2\bigr)\bigr\rangle$$

is isotropic. Notice that we can modify c_1 and c_2 to any values of the forms $\langle c_1, d_1, -c_1 d_1\rangle$ and $\langle c_2, d_2, -c_2 d_2\rangle$ respectively. Thus it suffices to show that the form

$$\langle a_1, b_1\rangle \perp -a_1 b_1 \langle c_1, d_1, -c_1 d_1\rangle \perp \langle -a_2, -b_2\rangle \perp a_2 b_2 \langle c_2, d_2, -c_2 d_2\rangle$$

is isotropic. But the last form is 10-dimensional, belongs to $I^2(F)$ and its Clifford invariant is equal to $(a_1, b_1) + (c_1, d_1) + (a_2, b_2) + (c_2, d_2) = 0$. In particular, this form belongs to $I^3(F)$ [P]. Since any 10-dimensional form from $I^3(F)$ is isotropic [P], we are done.

An example of two decompositions which cannot be connected by a chain of length 2 is as follows. Let k be a field, $a, b, c \in k^*$, $\langle\langle a, b, c\rangle\rangle \neq 0$, $(a, b)_{k(\sqrt{c})} \neq 0$, $F = k((t))$, $A = (a, b) + (c, t) = (c, t) + (a, b)$. Suppose that these decompositions are connected by a chain of length at most 2. Then the index of $(a, b) + (c, t) + (c', x^2 - a'c'y^2)$ is at most 2 for some $x, y \in F$, $a' \in D(\langle a, b, -ab\rangle)$, $c' \in D(\langle c, t, -ct\rangle)$, where, as usual, by $D(\varphi)$ we denote the set of nonzero values of the quadratic form φ. Obviously, we may assume that c' equals either c, or t, or $-ct$. We will consider these cases one by one.

(i) Assume $c' = c$. The condition $(a, b)_{k(\sqrt{c})} \neq 0$ is equivalent to the form $\langle a, b, -ab, -c\rangle$ being anisotropic. Suppose $x, y \in F$, and either $x \neq 0$, or $y \neq 0$. Then $x^2 - a'cy^2 \in k^* F^{*2}$, hence $(a, b) + (c, t) + (c, x^2 - a'cy^2) = (a, b) + (c, et)$ for some $e \in k^*$. Since $(a, b)_{k(\sqrt{c})} \neq 0$, and $c \notin k^{*2}$ (for $\langle\langle a, b, c\rangle\rangle \neq 0$), we get by Prop. 2.4 in [T] that $\operatorname{ind}(a, b) \otimes (c, et) = 4$, a contradiction.

(ii) Assume $c' = t$. Obviously, $x^2 - a'ty^2 \in F^{*2} \cup -a'tF^{*2}$, hence $(a, b) + (c, t) + (t, x^2 - a'ty^2)$ equals either $(a, b) + (c, t)$, or $(a, b) + (a'c, t)$. If the index of the last algebra is 2, then again by Prop. 2.4 of [T] either $a'c \in D(\langle a, b, -ab\rangle)$, or $a'c \in k^{*2}$, which implies that $c \in D(\langle\langle a, b\rangle\rangle)$, a contradiction in view of the hypothesis $\langle\langle a, b, c\rangle\rangle \neq 0$.

(iii) The case $c' = -ct$ is quite similar to case (ii). The algebra $(a, b) + (c, t) + (-ct, x^2 + a'cty^2)$ equals either $(a, b) + (c, t)$, or $(a, b) + (c, t) + (-ct, a') = (a, b) + (-c, a') + (a'c, t)$. If the index of the last algebra is 2, then as in case (ii) $a'c \in D(\langle a, b, -ab\rangle)$, or $a'c \in k^{*2}$, which is impossible. $\square$

Remark. Notice that in the above example $\mathrm{cd}_2 F \geqslant 4$. We do not know, however, whether any two decompositions can be connected by a chain of length 2 if $\mathrm{cd}_2 F \leqslant 3$.

Now we introduce another equivalence between biquaternion decompositions. Namely, we say that two decompositions $A = D_1 + D'_1 = D_2 + D'_2$ are strongly simply-equivalent if there exist elements $x, a, c \in F$ such that $D_{1F(\sqrt{a})} = D'_{1F(\sqrt{c})} = 0$ and $D_2 = D_1 + (a, c)$, $D'_2 = D'_1 + (a, c)$. Clearly, if the decompositions are strongly simply equivalent, they are simply equivalent (one can put $x = 0$, $y = 1$). As earlier we call two decompositions strongly equivalent if they can be connected by a chain of decompositions where any two neighboring decompositions are strongly simply-equivalent. The following statement shows that in fact there is no difference between equivalence and strong equivalence.

Proposition 2. *Any two biquaternion decompositions of A are strongly equivalent to one another, and can be connected by a chain of length* 6.

Proof. The proof is clear in view of the chain of length 2, connecting two simply-equivalent decompositions

$$\begin{aligned}&(a,b) + (c,d) \sim (ac,b) + (c,bd)\\&\quad = \bigl(ac, b\bigl(x^2 - acy^2\bigr)\bigr) + (c, bd) \sim \bigl(a, b\bigl(x^2 - acy^2\bigr)\bigr) + \bigl(c, d\bigl(x^2 - acy^2\bigr)\bigr),\end{aligned}$$

where each of the two steps is a strongly simply-equivalence. □

Recall that any biquaternion decomposition of A determines the corresponding Albert quadratic form, namely, the Albert form of the decomposition $A \simeq (a,b) \otimes (c,d)$ is $\langle a, b, -ab, -c, -d, cd\rangle$. As a consequence of Proposition 2 we obtain a strengthening of a well-known theorem of Jacobson on similarity of any two Albert forms of a biquaternion algebra [J,L].

Corollary 3. *Any two Albert forms of the same biquaternion algebra A are similar, and the coefficient of similarity can be chosen as a product of some $u_i \in F^*$ $(1 \leqslant i \leqslant 6)$, where $F(\sqrt{u_i})$ is a quadratic subalgebra of A.*

Proof. It is easy to find a similarity coefficient for two Albert forms corresponding to strongly simply-equivalent decompositions. Namely, let $A = (a,b) + (c,d) = (a,bc) + (c,ad)$. Then

$$\langle a, bc, -abc, -c, -ad, acd\rangle \simeq -ac\langle a, b, -ab, -c, -d, cd\rangle.$$

Since $F(\sqrt{ac})$ is a quadratic subalgebra of A, the corollary follows from Proposition 2. □

Remark. Proposition 1 can be restated as follows. Call two pairs of quaternion algebras (D_i, D'_i), $i = 1, 2$ *equal* if $D_1 \simeq D_2$ and $D'_1 \simeq D'_2$, and *simply equivalent* if there exist $a, b, c, d \in F^*$ and $x \in D(\langle\langle ac\rangle\rangle)$ such that $(D_1, D'_1) = ((a,b),(c,d))$, $(D_2, D'_2) = ((a,bx),(c,dx))$. Two pairs are called *equivalent* if there is a chain of simply equivalent pairs connecting them. Proposition 1 then reads:

> *Two pairs (D_i, D'_i), $i = 1, 2$ are equivalent iff $D_1 \otimes_F D'_1 \simeq D_2 \otimes_F D'_2$. In this situation, one can always find a chain of length* 3 *connecting the pairs, but generally not of shorter length.*

Proposition 2 can be restated in a similar way.

So far we have considered decompositions of biquaternion algebras into a sum of two quaternion algebras. Now we consider a similar problem, but this time decompositions into a sum of three quaternions. Let A be a central simple algebra of index 1, 2 or 4 over a field F of characteristic different from 2. Let further $A = D_1 + D'_1 + D''_1 = D_2 + D'_2 + D''_2$ be two decompositions of A into a sum of three quaternion algebras. We call these decompositions equal if $D_1 = D_2$, $D_1' = D_2'$, and $D_1'' = D_2''$, and simply-equivalent if there are $1 \leqslant i < j \leqslant 3$ such that the sums of the ith and jth summands in both decompositions are simply equivalent in the previous sense (in particular, the remaining summands are equal). As earlier we say that these decompositions are equivalent if they can be connected by a chain of decompositions in such a way that every two neighboring decompositions in this chain are simply-equivalent. Unfortunately, we are not able to prove that any two decompositions are equivalent in full generality, restricting ourselves to the following

Proposition 4.

(1) *If* $\operatorname{ind} A \leqslant 2$, *then any two decompositions of* A *into a sum of three quaternions are equivalent.*
(2) *If* $\operatorname{ind} A = 4$ *and* F *has no cubic extension, then any two decompositions of* A *into a sum of three quaternions are equivalent.*

Proof. (1) Let $D_1 + D_1' + D_1'' = Q$, where Q is a quaternion algebra. Then $D_1 + D_1' = Q + D_1''$. By Proposition 1 we have $D_1 + D_1' \sim Q + D_1''$. Hence

$$D_1 + D_1' + D_1'' \sim Q + D_1'' + D_1'' \sim Q + 0 + 0,$$

which shows that any two decompositions of Q are equivalent.

(2) Since the elements $D_1 + D_1'$ and A differ by a quaternion, there exists a field extension L/F of degree 4 such that $(D_1 + D_1')_L = A_L = 0$ [R]. Since F has no cubic extension, there exists an intermediate quadratic extension $F(\sqrt{a})$ between F and L. In particular, $\operatorname{ind}(D_1 + D_1')_{F(\sqrt{a})} \leqslant 2$ and $\operatorname{ind} A_{F(\sqrt{a})} \leqslant 2$. Hence

$$D_1 + D_1' = (a, b) + (c, d)$$

for some $b, c, d \in F^*$. Since $((c, d) + D_1'')_{F(\sqrt{a})} = A_{F(\sqrt{a})}$ is a quaternion algebra over $F(\sqrt{a})$, we get $(c, d) + D_1'' = (a, u) + (v, w)$ for some $u, v, w \in F^*$. Therefore,

$$D_1 + D_1' + D_1'' \sim (a, b) + (c, d) + D_1'' \sim (a, b) + (a, u) + (v, w) \sim 0 + (a, bu) + (v, w).$$

Since all decompositions of A into a sum of two quaternions are equivalent, we are done. □

Open questions. Is it true that any two biquaternion decompositions in Proposition 2 can be connected by a chain of length shorter than 6? Can one drop the hypothesis on the absence of cubic extensions of F in part (2) of Proposition 4? What is the situation in the case $\operatorname{ind} A = 8$?

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